Two-dimensional planing at high Froude number

By E. CUMBERBATCH
Department of Mathematics, University of Manchester

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SUMMARY

This paper examines the flow characteristics of a body of small slope planing at high Froude number over a water surface. An equation is obtained relating the slope of the planing surface to an integral containing the pressure distribution on the planing surface. The equation is expanded for large Froude number and a solution is obtained by an iteration process. At each stage of the iteration process the integral equation of ordinary thin aerofoil theory is solved. The pressure distribution on the planing surface is derived as a series in inverse powers of the Froude number F, as far as the F^{-4} term. Computations are performed for the planing of a flat plate, a parabolic surface, and a suitable linear combination of these shapes which results in a flow without a splash at the leading edge.

1. INTRODUCTION

When a surface craft moves through water at low speeds it remains near its floating position relative to the undisturbed water surface, the lift being supplied mainly by hydrostatic forces. If the speed of the craft is increased through a certain critical value, the craft rises and moves over the water, with separation of the flow taking place at the trailing edge. The craft is then supported by hydrodynamic forces and is said to be planing.

One of the features of a planing motion is the narrow jet of water, known as the splash, which is thrust forward by the planing surface near its leading edge. Neglecting friction, the drag force on a planing craft is made up of two parts: (1) a splash drag due to the work done in creating the splash, and (2) a wave drag representing the energy carried downstream in the wave motion set up by the passage of the craft. If the craft, which is taken to be of length $2l$, has a velocity U, waves fixed with respect to the craft are of wavelength $\lambda = 2\pi F^2 l$, where $F = U(gl)^{-1/2}$ is the Froude number.

This paper is concerned with the determination of the two-dimensional flow past planing surfaces which are inclined at small angles to the undisturbed water surface. A solution applicable at high Froude number is obtained for a planing surface of arbitrary shape. It is seen that the surface shape can be chosen to give a smooth flow at the leading edge, thereby eliminating the splash drag. Since the wave drag is small at high Froude number, the shape which eliminates the splash drag may provide a good approximation to the optimum design for minimum drag.

The solution in the infinite Froude number case has been given by Wagner (1932). The equations governing the flow are shown to be the same as in the flow past infinitely thin aerofoils. Except for the splash, which is assumed thin, the flow set up by the planing surface is identical to the flow in the lower half-plane of an unbounded fluid disturbed by an infinitely thin aerofoil of the same shape as the planing surface. The pressure distribution on the planing surface is the same as that on the lower side of the corresponding aerofoil and the lift is therefore half that of the aerofoil. An account of the method used by Wagner for obtaining the splash drag from the aerofoil solution is given in § 3. Wagner examines the flow past a body of circular camber at various angles of incidence of the chord line. At zero incidence drag-free planing is achieved since the wave drag is zero at infinite Froude number and for a symmetrical surface there is no splash drag since the flow must be smoothly joined to the planing surface at the leading edge as well as the trailing edge.

Lamb (1932, §§ 242–4) considers the two-dimensional flow due to the application of a pressure point to the surface of a stream. The planing of a body at arbitrary Froude number can be represented by a distribution of pressure points whose strength is proportional to the excess pressure on the body at each point. The boundary condition of zero normal velocity on the body then leads to an equation relating the slope of the planing surface to an integral involving the pressure distribution along the planing surface. Lamb considers two simple pressure distributions for which the integrals can be evaluated to give the shape of the planing surface.

Two attacks on the integral equation involved in the inverse problem of finding the pressure distribution on the planing surface in terms of the given surface shape have been made by Maruo (1951) and Squire (1957). Maruo assumes a Fourier series expansion for the pressure distribution in the form

$$p = a_0 \frac{1-\cos\theta}{\sin\theta} + \sum_{n=1}^{\infty} a_n \sin n\theta, \tag{1}$$

where $x = -l\cos\theta$ for $0 \leqslant \theta \leqslant \pi$ denotes the distance measured from the centre of the planing surface. An infinite series for the body slope results, comprising an infinite set of functions multiplying the coefficients a_n. From this set an orthogonal set is constructed, which enables equations for the a_n's to be derived. The example of the planing of a flat plate at general Froude number is computed, yielding the pressure distribution and the lift and drag coefficients as functions of the Froude number. A diagram is given showing the variation of the constants a_0 to a_5 with Froude number, but no other indication is given regarding how many constants have to be calculated for accurate results. Maruo's work provides the solution to the planing problem of an arbitrary surface shape at general Froude number. However, the analysis necessary in obtaining and using the orthogonal functions described above is very complicated and the present paper is intended to provide a simple alternative to this method.

Squire considers only the first four terms in a similar series for the pressure and calculates the coefficients to give a surface shape as nearly flat as possible by imposing mean curvature conditions. The results obtained agree over a wide range of Froude numbers with those given by Maruo for the flat plate.

A new approach is presented in § 2 of this paper. This enables the solution for an arbitrary body shape to be easily computed, provided the Froude number is moderately high. It is seen, from comparison with the pressure distributions on a flat plate given by Maruo, that the approximate solution derived here provides accurate results for Froude numbers greater than 3.

The equation relating the planing surface slope to an integral containing the pressure distribution on the planing surface is solved by an iteration process. The integral is expanded as a series in inverse powers of the Froude number. An iteration process provides successive terms in the pressure distribution expansion in a similar series. At infinite Froude number, the solution to the integral equation is the thin aerofoil solution, yielding the first term in the pressure distribution expansion as an integral of the body slope. At each stage of the iteration procedure the same thin aerofoil theory integral equation has to be solved. The solution of this equation provides each new term in the expansion of the pressure distribution as an integral of the previous terms. The expansion up to terms of order F^{-4} is obtained in § 3 and the multiple integrals obtained in the solution are reduced to single integrals of the body slope, involving certain functions which are given in tables 1 and 2. The lift and drag coefficients are obtained in similar form.

The large Froude number approximations for the planing of a flat plate and of a parabolic camber shape are given in § 4. As already mentioned, a parabolic surface at zero incidence, planing at infinite Froude number, has smooth flow at the leading edge and therefore has no splash drag. An important result, that emerges from the study of the flat plate and parabolic camber flows at high Froude number, is that there always exists a linear combination of the two flows which represents a flow having no splash. In this high Froude number range, where the wave drag is small, elimination of the splash drag causes a considerable reduction in total drag. The planing characteristics for a body shape derived in this manner are discussed in § 4.

2. Large Froude number approximation

A surface craft, having large span so that the motion can be assumed two-dimensional, is considered to be planing at a speed U over a water surface. The water is considered to be incompressible and of infinite depth. The planing surface is assumed to be inclined at a small angle to the horizontal at all points of its length. Cartesian coordinates are taken with the origin at the middle of the planing surface, the x-axis in the direction of motion, the y-axis vertically upwards and half the length of the wetted surface is taken as the unit of length.

For an irrotational motion, a solution to Laplace's equation is sought in the lower half-plane which satisfies the boundary conditions on the free surface and on the body. The boundary conditions are linearized with respect to the perturbation velocities and are applied on $y = 0$, since the height of the water surface,

$$y = \eta(x), \tag{2}$$

is of the same order as the perturbation velocities.

The linearized boundary condition of zero normal flow across the water surface is

$$\frac{d\eta}{dx} = -\frac{\partial \phi}{\partial y}, \tag{3}$$

where $U\nabla\phi(x,y)$ is the perturbation velocity. On the planing surface $|x| \leqslant 1$, $d\eta/dx$ is prescribed as $S(x)$, the slope of the surface. Bernoulli's equation gives a second relation between η and ϕ to be applied on $y = 0$ as

$$\frac{p}{\rho U^2} - \frac{\partial \phi}{\partial x} + K\eta + \mu\phi = 0, \tag{4}$$

where $K = F^{-2}$ and ρ is the density of water. The pressure p, measured in excess of atmospheric pressure, is zero on the free surface and is to be determined on the planing surface. The term $\mu\phi$ in equation (4) represents Rayleigh's artificial friction, which ensures no disturbance far ahead of the craft. The term can alternatively be regarded (according to Laplace transform theory) as the contribution for large time from $\partial\phi/\partial t$, for the unsteady problem of a craft starting from rest. The required steady solution is obtained by eventually letting μ tend to zero through positive values.

A potential which satisfies the free surface conditions is given by Lamb, the planing surface being represented by a distribution of pressure points, and may be written as

$$\phi = -\lim_{\mu\to 0} \mathscr{R}\frac{i}{\pi}\int_{-1}^{1} P(\xi)\left[\int_0^\infty \frac{e^{i\alpha(x-\xi)+\alpha y}}{\alpha - K + i\mu}\,d\alpha\right] d\xi, \tag{5}$$

where $p(\xi) = \rho U^2 P(\xi)$ is the excess pressure on the body. The boundary condition (3) on the planing surface leads to the integral equation for the pressure distribution

$$S(x) = \frac{1}{\pi}\int_{-1}^{1} P(\xi)\frac{d\xi}{\xi - x} + \lim_{\mu\to 0}\mathscr{R}\frac{iK}{\pi}\int_{-1}^{1} P(\xi)\left[\int_0^\infty \frac{e^{i\alpha(x-\xi)}}{\alpha - K + i\mu}\,d\alpha\right] d\xi, \tag{6}$$

where the Cauchy principal value is to be taken in the first integration on the right-hand side.

The inner integral in the last term of equation (6) will now be transformed into the exponential integral by putting

$$x - \xi = a, \qquad ia(\alpha - K + i\mu) = X. \tag{7}$$

The integral with respect to X parallel to the imaginary axis is deformed into an integral parallel to the real axis, as shown in figure 1, there being no contribution from paths at infinity in the negative real half-plane.

On letting $\mu \to 0$ through positive values,

$$\lim_{\mu\to 0}\int_0^\infty \frac{e^{i\alpha(x-\xi)}}{\alpha - K + i\mu}\,d\alpha = -e^{iaK}\left(2\pi i H(-a) + \int_{-\infty}^{-iaK}\frac{e^X}{X}\,dX\right), \tag{8}$$

where $H(x)$ is the Heaviside unit function. The residue term $-2\pi i e^{iaK}$ is included when $a < 0$ since the contour is then deformed over the pole at $X = 0$. The expansion for small K of the integral on the right-hand side of equation (8) is given by Jahnke & Emde (1945), enabling equation (6) to be written

$$S(x) = \frac{1}{\pi}\int_{-1}^{1} P(\xi)\frac{d\xi}{\xi - x} + \\ + \frac{K}{\pi}\int_{-1}^{1} P(\xi)[\pi - \tfrac{1}{2}\pi \operatorname{sgn}(x-\xi) + (\gamma - 1)K(x-\xi) + \\ + K(x-\xi)\log(K|x-\xi|) + \ldots]\, d\xi, \quad (9)$$

where $\gamma = 0{\cdot}5772$ is Euler's constant. This equation is solved by an iteration process on the small parameter K, yielding the pressure $P(\xi)$ as a series in K.

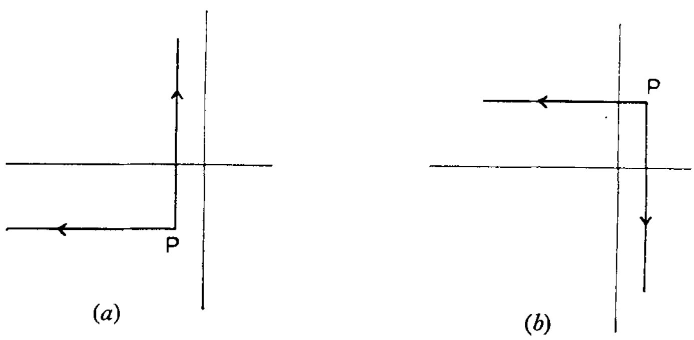

Figure 1. Integral paths for the exponential integral in equation (6), travelling from the point $P = -ia(K - i\mu)$ parallel to the imaginary axis, are deformed into paths parallel to the real axis as indicated in (a) for $a > 0$ and in (b) for $a < 0$. The contour is deformed over the pole at $X = 0$ in (b).

In the infinite Froude number case the integral equation (9) for the pressure distribution on the body reduces to

$$S(x) = \frac{1}{\pi}\int_{-1}^{1} P(\xi)\frac{d\xi}{\xi - x}. \quad (10)$$

This equation is that of thin aerofoil theory and is discussed, for example, by Tricomi (1957). The solution to equation (10) gives the first term, $P_0(\xi)$, in the expansion of the pressure, as

$$P_0(\xi) = -\frac{1}{\pi}\sqrt{\left(\frac{1+\xi}{1-\xi}\right)}\int_{-1}^{1}\frac{S(t)}{t-\xi}\sqrt{\left(\frac{1-t}{1+t}\right)}\,dt. \quad (11)$$

Equation (10) admits another solution of similar form which is rejected on account of the Kutta condition that the singularity has to be at the leading edge. Using the solution for $P_0(\xi)$, equation (9) when approximated to the first order in K becomes

$$S(x) - K\int_{-1}^{1} P_0(\xi)[1 - \tfrac{1}{2}\operatorname{sgn}(x-\xi)]\,d\xi = \frac{1}{\pi}\int_{-1}^{1} P(\xi)\frac{d\xi}{\xi - x}, \quad (12)$$

where the known functions have been collected on the left-hand side. This integral equation has the same form as the aerofoil integral equation. A solution for the second approximation $P_0(\xi)+KP_1(\xi)$ follows, where $P_1(\xi)$ is given by

$$P_1(\xi) = \frac{1}{\pi}\sqrt{\left(\frac{1+\xi}{1-\xi}\right)}\int_{-1}^{1}\left[\int_{-1}^{t} P_0(r)(1-\tfrac{1}{2}\operatorname{sgn}(t-r))\,dr\right]\sqrt{\left(\frac{1-t}{1+t}\right)}\frac{dt}{t-\xi}. \quad (13)$$

Equation (9) may now be approximated to the second order in K using the above solution for $P_0(\xi)+KP_1(\xi)$ and the third approximation

$$P_0(\xi)+KP_1(\xi)+K^2P_2(\xi)$$

to the pressure distribution follows in like manner. $P_2(\xi)$ is given by

$$P_2(\xi) = \frac{1}{\pi}\sqrt{\left(\frac{1+\xi}{1-\xi}\right)}\int_{-1}^{1}\left[\int_{-1}^{t} Q(r,t)\,dr\right]\sqrt{\left(\frac{1-t}{1+t}\right)}\frac{dt}{t-\xi}, \quad (14)$$

where

$$Q(r,t) = P_1(r)(1-\tfrac{1}{2}\operatorname{sgn}(t-r)) + \\ + \frac{1}{\pi}P_0(r)[(\gamma-1)(t-r)+(t-r)\log(K|t-r|)].$$

The expressions (11), (13) and (14) provide the expansion of the pressure distribution on the body up to terms of the order of F^{-4} in the form of integrals involving previous terms in the series. In the following section these integrals are transformed by successive substitutions to ones containing the given function $S(t)$ only. The resulting multiple integrals are reduced to single integrals.

3. Simplification of the solution: lift and drag

In the expressions for the terms in the expansion of the pressure distribution on the body given in the previous section, the distance integrations along the body can be replaced by angular integrations by substitutions of the form

$$\xi = -\cos\phi \quad \text{for } 0 \leqslant \phi \leqslant \pi. \quad (15)$$

Equation (11) for the first term in the pressure distribution expansion may then be rewritten

$$P_0(-\cos\phi) = A_0\tan\tfrac{1}{2}\phi + \frac{1}{\pi}\sin\phi\int_0^{\pi} S(-\cos\theta)\,\frac{d\theta}{\cos\theta-\cos\phi}, \quad (16)$$

where

$$A_0 = \frac{1}{\pi}\int_0^{\pi} S(-\cos\theta)\,d\theta.$$

Substituting this expression for $P_0(-\cos\phi)$ in equation (13) for the second term in the expansion leads to

$$P_1(-\cos\phi) = A_1\tan\tfrac{1}{2}\phi + B_1\,T(\phi) - \frac{1}{\pi^2}\int_0^{\phi}\left[\int_0^{\pi}\frac{\sin\theta}{\cos\theta-\cos\psi}\times \\ \times\left(\int_0^{\pi}\frac{S(-\cos\gamma)\sin^2\gamma\,d\gamma}{\cos\theta-\cos\gamma}\right)d\theta\right]d\psi, \quad (17)$$

where
$$A_1 = \tfrac{1}{2}\pi B_1 - \frac{2}{\pi}\int_0^\pi S(-\cos\theta)\sin\theta\, T(\theta)\, d\theta,$$
$$B_1 = -\frac{2}{\pi}\int_0^\pi S(-\cos\theta)(1+\cos\theta)\, d\theta,$$
$$T(\phi) = -\frac{1}{\pi}\int_0^\phi \log\tan\tfrac{1}{2}\theta\, d\theta.$$

Values of $T(\phi)$ are given in table 1. This form of $P_1(-\cos\phi)$ is convenient for some very simple $S(-\cos\phi)$ since the formula
$$\int_0^\pi \frac{\cos n\theta}{\cos\theta - \cos\phi}\, d\theta = \frac{\pi \sin n\phi}{\sin\phi} \tag{18}$$
enables the multiple integral in (17) to be evaluated directly. For general

ϕ	$T(\phi)$	$Z(\phi)$	ϕ	$T(\phi)$	$Z(\phi)$
0	0	0	26	0·3579	0·1369
1	0·0319	0·0355	30	0·3887	0·1345
2	0·0561	0·0541	34	0·4165	0·1305
3	0·0774	0·0678	38	0·4415	0·1251
4	0·0968	0·0788	42	0·4640	0·1185
5	0·1148	0·0878	46	0·4841	0·1111
6	0·1316	0·0954	50	0·5021	0·1030
7	0·1476	0·1019	54	0·5181	0·0942
8	0·1627	0·1075	58	0·5321	0·0849
9	0·1772	0·1124	62	0·5443	0·0751
10	0·1910	0·1166	66	0·5548	0·0650
12	0·2171	0·1234	70	0·5635	0·0546
14	0·2412	0·1284	74	0·5706	0·0440
16	0·2637	0·1321	78	0·5761	0·0331
18	0·2848	0·1347	82	0·5800	0·0222
20	0·3047	0·1364	86	0·5823	0·0111
22	0·3234	0·1372	90	0·5831	0

Table 1. Values of the functions $T(\phi)$ and $Z(\phi)$ defined in equations (17) and (21), with the argument ϕ measured in degrees. For the range $90^\circ \leqslant \phi \leqslant 180^\circ$, the formulae $T(\phi) = T(180-\phi)$, $Z(\phi) = -Z(180-\phi)$ may be used. The second of these relations follows from the result $\int_0^\pi \log^2\tan\tfrac{1}{2}\phi\, d\phi = \tfrac{1}{4}\pi^3$.

body slope the multiple integral in (17) is reduced to a single γ-integration suitable for numerical computation by first performing the θ- and ψ-integrations. The changing of the order of integration requires caution because of the double pole in the integrand when $\theta = \gamma = \psi$. The identity
$$\int_0^\pi \frac{\sin\theta}{\cos\theta-\cos\psi}\left[\int_0^\pi \frac{S(-\cos\gamma)\sin^2\gamma}{\cos\theta-\cos\gamma}\, d\gamma\right] d\theta$$
$$= \int_0^\pi \frac{\sin\theta}{\cos\theta-\cos\psi}\left[\int_0^\pi \frac{(S(-\cos\gamma)-S(-\cos\psi))\sin^2\gamma}{\cos\theta-\cos\gamma}\, d\gamma\right] d\theta +$$
$$+ S(-\cos\psi)\int_0^\pi \frac{\sin\theta}{\cos\theta-\cos\psi}\left[\int_0^\pi \frac{\sin^2\gamma}{\cos\theta-\cos\gamma}\, d\gamma\right] d\theta \tag{19}$$

overcomes this difficulty since the order of integration in the first term on the right-hand side may now be changed as it does not contain the double pole. The second term in (19) may be evaluated directly since the integrations do not involve the body slope. Performing the θ- and ψ-integrations and making the substitutions $t = \tan\frac{1}{2}\psi(\tan\frac{1}{2}\gamma)^{-1}$ and $u = \tan\frac{1}{2}\phi(\tan\frac{1}{2}\gamma)^{-1}$, equation (17) is reduced to the form

$$P_1(-\cos\phi) = A_1\tan\tfrac{1}{2}\phi + B_1\,T(\phi) + \frac{16}{\pi^2}\tan^2\tfrac{1}{2}\phi\int_0^1 u\,F_1(u)\left[S\left(\frac{\tan^2\frac{1}{2}\phi - u^2}{\tan^2\frac{1}{2}\phi + u^2}\right)(u^2+\tan^2\tfrac{1}{2}\phi)^{-2} - S\left(\frac{u^2\tan^2\frac{1}{2}\phi - 1}{u^2\tan^2\frac{1}{2}\phi + 1}\right)(u^2\tan^2\tfrac{1}{2}\phi + 1)^{-2}\right]du, \quad (20)$$

where

$$F_1(x) = \int_0^x \frac{\log t}{t^2-1}\,dt.$$

Values of $F_1(x)$ are given in table 2.

x	$F_1(x)$	$-F_2(x)$	x	$F_1(x)$	$-F_2(x)$
0	0	0	0·16	0·4562	1·4504
0·001	0·0079	0·0635	0·18	0·4927	1·5151
0·002	0·0144	0·1061	0·20	0·5272	1·5724
0·004	0·0261	0·1741	0·22	0·5598	1·6234
0·006	0·0367	0·2304	0·24	0·5909	1·6691
0·008	0·0466	0·2798	0·26	0·6205	1·7101
0·010	0·0561	0·3242	0·28	0·6487	1·7527
0·015	0·0780	0·4206	0·30	0·6758	1·7806
0·02	0·0983	0·5026	0·35	0·7386	1·8514
0·025	0·1172	0·5747	0·40	0·7957	1·9075
0·030	0·1352	0·6394	0·45	0·8480	1·9523
0·035	0·1524	0·6982	0·50	0·8961	1·9881
0·040	0·1688	0·7522	0·55	0·9406	2·0168
0·045	0·1847	0·8022	0·60	0·9819	2·0398
0·050	0·1999	0·8488	0·65	1·0205	2·0597
0·06	0·2290	0·9332	0·70	1·0566	2·0721
0·07	0·2565	1·0084	0·75	1·0905	2·0831
0·08	0·2825	1·0759	0·80	1·1224	2·0912
0·09	0·3074	1·1371	0·85	1·1526	2·0970
0·10	0·3311	1·1931	0·90	1·1811	2·1008
0·12	0·3759	1·2919	0·95	1·2081	2·1023
0·14	0·4174	1·3767	1·00	1·2337	2·1036

Table 2. $F_1(x)$, $F_2(x)$ are defined in equations (20) and (21). The relations $F_1(1) = \frac{1}{8}\pi^2$, $F_2(1) = -\frac{1}{2}\pi\int_0^\pi T(\phi)\,d\phi$, provide a check on the accuracy of tables 1 and 2.

$P_2(-\cos\phi)$ may be similarly reduced to single integrals involving the body slope. The logarithmic term in equation (14) may be dealt with by integrating equation (10) from -1 to y with respect to x and then from -1 to r with respect to y. Using techniques similar to those applied in reducing $P_1(-\cos\phi)$, $P_2(-\cos\phi)$ may be expressed as

$$\begin{aligned}P_2(-\cos\phi) = {} & A_2\tan\tfrac{1}{2}\phi + B_2\,T(\phi) + C_2\sin\phi - (2A_1/\pi)\sin\phi\log\tan\tfrac{1}{2}\phi - \\ & -(B_1/\pi^2)\sin\phi\log^2\tan\tfrac{1}{2}\phi + \\ & + \left(\tfrac{1}{2}\pi B_1\cos\phi - \int_0^\pi S(-\cos\theta)\sin^2\theta\,d\theta\right)Z(\phi) - \\ & - \frac{16}{\pi^3}\sin\phi\tan^3\tfrac{1}{2}\phi\int_0^1 (u-u^3)F_2(u)\times \\ & \times\left[S\left(\frac{\tan^2\frac{1}{2}\phi - u^2}{\tan^2\frac{1}{2}\phi + u^2}\right)(u^2+\tan^2\tfrac{1}{2}\phi)^{-3} - \right. \\ & \left. - S\left(\frac{u^2\tan^2\frac{1}{2}\phi - 1}{u^2\tan^2\frac{1}{2}\phi + 1}\right)(u^2\tan^2\tfrac{1}{2}\phi+1)^{-3}\right]du, \qquad (21)\end{aligned}$$

where

$$\begin{aligned}A_2 &= -\frac{2A_1}{\pi} + \frac{\pi B_2}{2} + \int_0^\pi S(-\cos\theta)Y(\theta)\,d\theta, \\ B_2 &= -2A_1 - \frac{2B_1}{\pi} - \frac{4}{\pi^2}\int_0^\pi S(-\cos\theta)\sin^2\theta\log\tan\tfrac{1}{2}\theta\,d\theta, \\ C_2 &= -\tfrac{1}{2}\pi L B_1 + \int_0^\pi S(-\cos\theta)\sin\theta\,Z(\theta)\,d\theta, \\ L &= \frac{1}{2\pi}\left(2\gamma - 1 + 2\log\tfrac{1}{2}K + \frac{4}{\pi}\int_0^\pi T(\theta)\,d\theta\right), \\ Y(\theta) &= (2/\pi^3)\sin^2\theta\log^2\tan\tfrac{1}{2}\theta + L\sin^2\theta - \tfrac{1}{2}\sin 2\theta\,Z(\theta), \\ Z(\theta) &= \frac{2}{\pi^3}\int_0^\theta \log^2\tan\tfrac{1}{2}\gamma\,d\gamma - \frac{\theta}{2\pi}, \\ F_2(x) &= \int_0^x \frac{\log^2 t}{t^2-1}\,dt.\end{aligned}$$

Values of the functions $Z(\theta)$ and $F_2(x)$ are given in tables 1 and 2.

The lift coefficient,

$$C_L = \int_0^\pi P(-\cos\phi)\sin\phi\,d\phi,$$

may now be obtained to the second order in K as

$$\begin{aligned}C_L = {} & -\tfrac{1}{2}\pi B_1 - \tfrac{1}{2}\pi B_2 K + \left[\pi\left(A_2 - \frac{B_1}{8} + \frac{C_2}{2}\right) + \frac{B_2}{2} - A_1 + \frac{B_1}{2\pi} - \right. \\ & \left. - \int_0^\pi S(-\cos\theta)\left(\frac{1}{4} + \frac{1}{\pi^2}\log^2\tan\tfrac{1}{2}\theta\right)\sin^2\theta\cos\theta\,d\theta\right]K^2. \qquad (22)\end{aligned}$$

Similarly the drag coefficient,

$$C_D = \int_0^\pi P(-\cos\phi)S(-\cos\phi)\sin\phi\,d\phi,$$

to the second order in K is

$$C_D = \pi[A_0^2 + (2A_0A_1 + \tfrac{1}{4}\pi B_1^2)K + (2A_0A_2 + A_1^2 + \tfrac{1}{2}\pi B_1B_2)K^2]. \qquad (23)$$

The splash drag is determined by considering the singular term $A\tan\frac{1}{2}\phi$ in the pressure distribution on the body. Wagner interprets this term, which corresponds to a term with a $z^{1/2}$ behaviour in the complex potential, as describing the reversal of the fluid in the narrow jet of width δ forming the splash and he obtains the relation $\delta = \frac{1}{2}\pi A^2$. The work done to produce the splash may then be calculated in terms of A to give the splash drag coefficient as

$$C_S = \pi A^2. \tag{24}$$

To the second order in K, therefore,

$$C_S = \pi[A_0^2 + 2A_0 A_1 K + (2A_0 A_2 + A_1^2)K^2]. \tag{25}$$

The shape of the free surface is

$$\eta(x) = \int_{-1}^{1} P(\xi)(1-\operatorname{sgn}(x-\xi))\sin K(x-\xi)\, d\xi + \\ + \frac{1}{\pi}\int_{-1}^{1} P(\xi)[\{\tfrac{1}{2}\pi \operatorname{sgn}(x-\xi) - S_i(K(x-\xi))\}\sin K(x-\xi) - \\ - C_i(K(x-\xi))\cos K(x-\xi)]\, d\xi, \tag{26}$$

where S_i and C_i are the sine and cosine integrals. The wave drag is derived from the form of the waves far downstream. The first term on the right-hand side of the equation for $\eta(x)$ represents the wave train downstream, while the second term represents a local disturbance. The wave drag coefficient is deduced to be

$$C_W = K\left[\left(\int_{-1}^{1} P(\xi)\cos K\xi\, d\xi\right)^2 + \left(\int_{-1}^{1} P(\xi)\sin K\xi\, d\xi\right)^2\right] \\ = \tfrac{1}{4}\pi^2 B_1^2 K + \tfrac{1}{2}\pi^2 B_1 B_2 K^2, \tag{27}$$

expanded to the second order in K. It is noticed that the large Froude number approximations for the splash and wave drag given by (25) and (27) add up to give the total drag (23). The relation

$$C_W = KC_L^2 \tag{28}$$

valid to the second order in K, may be obtained from equations (22) and (27).

4. Large Froude number planing for flat and parabolic shapes

In this section the large Froude number expansion for the pressure distribution up to terms of order K^2 is derived for planing surfaces of flat and parabolic shape. For the planing of a flat plate the body slope is given by $S(-\cos\theta) = \alpha$, where α is the inclination of the plate. Substitution of this value of $S(-\cos\theta)$ in the expressions (16), (20) and (21) for the pressure distribution in the previous section gives

$$(1/\alpha)P(-\cos\phi) = A_F \tan\tfrac{1}{2}\phi + B_F\, T(\phi) + C_F \sin\phi - \\ - \left(\frac{2}{\pi}K - 2\left(1+\frac{1}{\pi^2}\right)K^2\right)\sin\phi \log\tan\tfrac{1}{2}\phi + \\ + (K^2/\pi^2)\sin\phi(2+\cos\phi)\log^2\tan\tfrac{1}{2}\phi + \\ + \tfrac{1}{8}K^2\sin 2\phi - \tfrac{1}{2}\pi K^2(1+2\cos\phi)Z(\phi), \tag{29}$$

where

$$A_F = 1-(\pi+2/\pi)K+\tfrac{1}{2}K^2\log\tfrac{1}{2}K+\left(\pi^2+6+\frac{1}{\pi^2}+\tfrac{1}{2}\gamma+\frac{1}{\pi}\int_0^\pi T(\theta)\,d\theta\right)K^2,$$

$$B_F = -2K+2(\pi+4/\pi)K^2,$$

$$C_F = K^2\log\tfrac{1}{2}K+\left(\gamma-\frac{1}{2}+\frac{2}{\pi}\int_0^\pi T(\theta)\,d\theta\right)K^2.$$

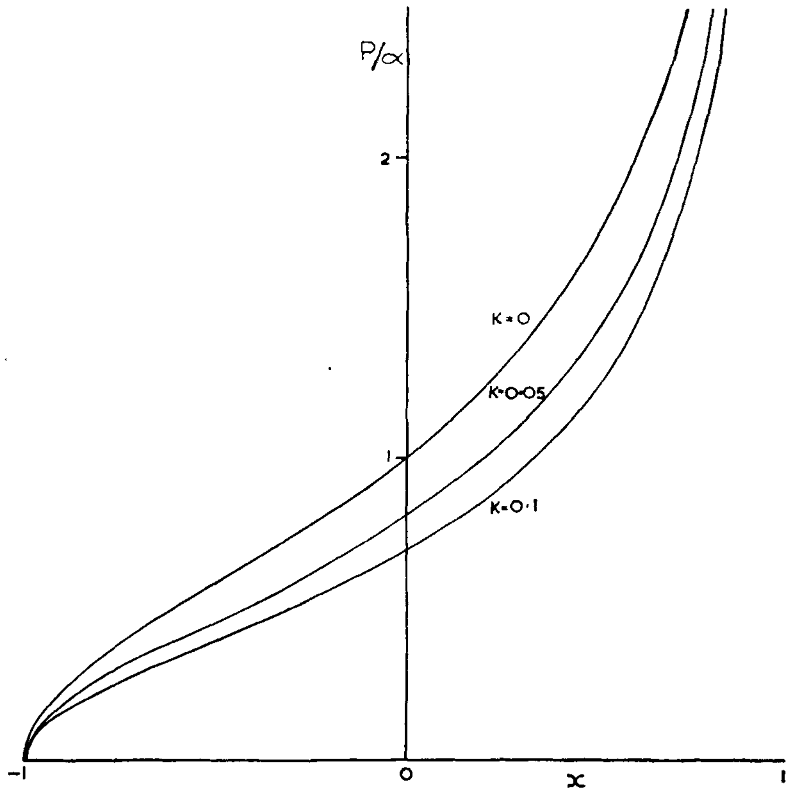

Figure 2. Distributions of the pressure coefficient P/α on a flat plate at inclination α, planing at Froude numbers corresponding to $K = 0, 0{\cdot}05$ and $0{\cdot}1$. The lift coefficients in these three cases are $C_L = \pi\alpha$, $0{\cdot}82\pi\alpha$ and $0{\cdot}73\pi\alpha$, respectively. The leading and trailing edges are at $x = 1$ and $x = -1$.

The pressure distributions on a flat plate planing at Froude numbers corresponding to $K = 0$, $0{\cdot}05$ and $0{\cdot}1$, are drawn in figure 2. Comparison with the exact pressure profiles on a flat plate obtained by Maruo show that the large Froude number expansion up to terms of order K^2 is adequate for K less than about $0{\cdot}1$. At $K = 0{\cdot}1$, which corresponds to a Froude number of about 3, there is a 6% discrepancy.

For a planing surface of parabolic camber the slope of the surface is taken to be $S(-\cos\theta) = \beta\cos\theta$. The pressure distribution expansion to terms of order K^2 for this shape is

$$(1/\beta)P(-\cos\phi) = A_p \tan\tfrac{1}{2}\phi + B_p\, T(\phi) + C_p \sin\phi - \left(\frac{K}{2\pi} - \frac{K^2}{3\pi^2}\right)\sin 2\phi \log\tan\tfrac{1}{2}\phi + K^2 \sin\phi \log\tan\tfrac{1}{2}\phi + \frac{K^2}{\pi^2}(1-\tfrac{1}{3}\sin^2\phi)\sin\phi\log^2\tan\tfrac{1}{2}\phi - \frac{K^2}{12}\sin^3\phi - \frac{\pi}{2}K^2\cos\phi\, Z(\phi), \quad (30)$$

where

$$A_p = -\tfrac{1}{2}\pi K + (\tfrac{1}{2}\pi^2 + \tfrac{7}{8})K^2,$$

$$B_p = -K + \left(\pi + \frac{8}{3\pi}\right)K^2,$$

$$C_p = 1 - \frac{K}{\pi} + \tfrac{1}{2}K^2\log\tfrac{1}{2}K + \left(\frac{\gamma}{2} - \frac{1}{4} + \frac{1}{3\pi^2} + \frac{1}{\pi}\int_0^\pi T(\theta)\,d\theta\right)K^2.$$

At infinite Froude number the singular term $A_p \tan\frac{1}{2}\phi$ in the pressure distribution disappears. This represents a flow which is smooth at the leading edge and the motion is achieved with zero splash drag. The pressure distribution in this case is elliptical.

From the pressure distributions for the flat plate and parabola, it can be seen how a flow without a splash drag may be obtained at any preassigned high Froude number. If the planing surface has a slope which is a linear combination of the slopes $S(-\cos\theta) = \alpha$ and $S(-\cos\theta) = \beta\cos\theta$, so chosen to eliminate the singular term $\tan\frac{1}{2}\phi$ in the resulting pressure distribution, then a splash-free flow will result. For the Froude number range $F > 3$, the pressure distribution on a planing surface chosen in this way approximates to the elliptical distribution on the parabolic shape planing at infinite Froude number. The required combination is obtained by choosing the constants α and β to satisfy the relation

$$\alpha : \beta = -A_p : A_F, \quad (31)$$

where A_F and A_p are defined in equations (29) and (30). It is to be emphasized that the splash-free flow obtained by satisfying this relation is attained at only one particular Froude number. As β determines the shape of the planing surface, the above relation is satisfied for any given parabolic shape by trimming the craft to give the required angle of incidence α.

The lift coefficient for a combination of the flat plate and parabola is derived by substituting $S(-\cos\theta) = \alpha + \beta\cos\theta$ in (22), and is given by

$$C_L = \alpha\pi(1 - 4{\cdot}41K + K^2\log\tfrac{1}{2}K + 19{\cdot}9K^2) + \tfrac{1}{2}\beta\pi(1 - 3{\cdot}99K + \tfrac{1}{2}K^2\log\tfrac{1}{2}K + 17{\cdot}4K^2). \quad (32)$$

This represents the sum of the lift coefficients for the flat and parabolic shapes (the pressure distribution for the combination is derived by adding the pressure distributions in (29) and (30)). Equations (31) and (32) may alternatively be used to express α and β in terms of C_L and K (correct to terms of order K^2) in the form

$$\alpha = KC_L, \tag{33}$$

$$\beta = (2/\pi)C_L(1+0{\cdot}85K - \tfrac{1}{2}K^2 \log \tfrac{1}{2}K - 0{\cdot}2K^2). \tag{34}$$

These equations give the necessary incidence α and shape (specified by β) to obtain a given lift coefficient for planing without splash at any particular Froude number $F > 3$. For this range of Froude numbers, corresponding to $K < 0{\cdot}1$, the K^2 terms in (34) are less than 2% of β, and both α and β may be taken as linear in K.

The shape of such a splash-free planing surface for a Froude number corresponding to $K = 0{\cdot}05$ is shown in figure 3, together with the local wave profile. Since the lift coefficient determines the magnification in the vertical direction it is convenient to plot $y(C_L)^{-1}$ instead of y. The shape of the free surface is obtained from equation (26) by numerical integration using the large Froude number approximation for the pressure distribution. The drag coefficient for the splash-free craft planing at $K = 0{\cdot}05$ is derived

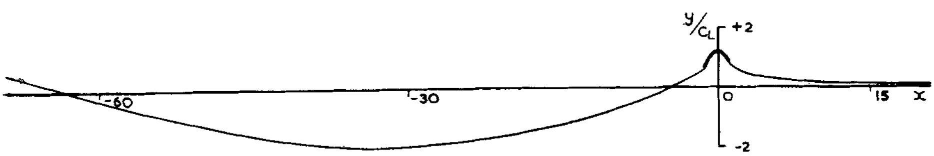

Figure 3. Splash-free craft planing at $K = 0{\cdot}05$ shown with local free surface shape. The craft lies between $x = -1$ and $x = +1$. Vertical scale is enlarged as indicated.

from (28) as $C_D = 0{\cdot}05\,C_L^2$. By way of comparison, the drag coefficient for a flat plate planing at the same Froude number is $C_D = 0{\cdot}39C_L^2$. This drag is obtained by adding the wave drag (28) to the splash drag $C_S = \pi A_F^2$ (24), where A_F is defined in terms of α by (29) and may be expressed in terms of C_L by using (32) with $\beta = 0$. A considerable reduction in drag is thus effected by eliminating the splash, so that the shape of such a splash-free craft may provide a good approximation to the optimum design for minimum drag.

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